Chaos in the Corridor

The School of Physics double pendulum

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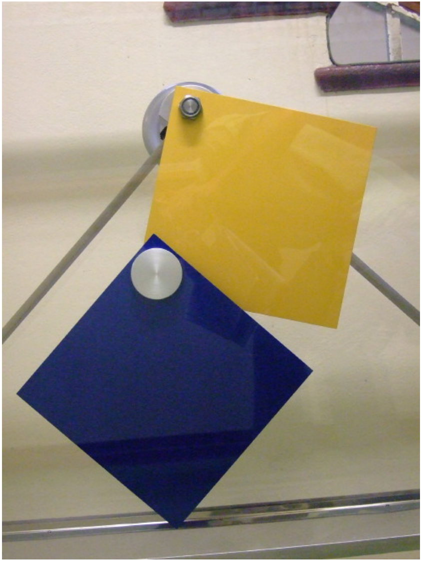
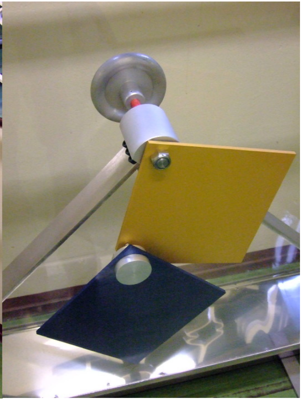


Overview

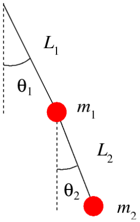
Background: The device

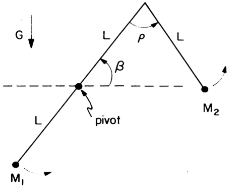
- Demonstration device in main corridor
  - built by School workshop in 1998
  - double square pendulum
  - plates about 28 cm on a side
  - set into motion by wheel at rear

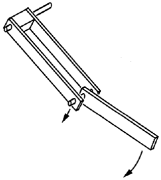



Double pendulums

- Simple double pendulum
  - textbook example for Lagrangian, Hamiltonian methods
    (e.g. Landau & Lifshitz 1976)
  - one of the simplest systems to exhibit chaos
    (e.g. Korsch & Jodl 1999)
  - asymmetrical versions considered
    (e.g. Newton 1989 – below centre)
- Real pendulums are compound (distributed mass)
  - double bar pendulum popular
    (e.g. Shinbrot, Gregobi, Wisdom, & Yorke 1992 – below right)

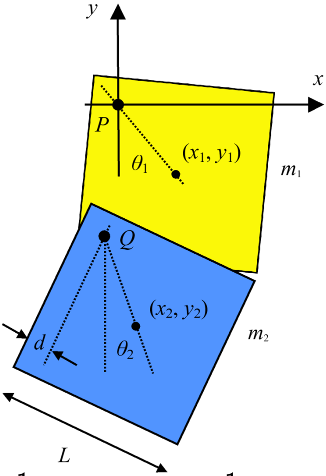






Modelling: Details

- Square plates, axles at P and Q
- Simplifications:
  - friction neglected
  - axles, wheel omitted
  - COM at centre of plate
  - $d = 0$, $m = m_1 = m_2$ used for numerical solutions



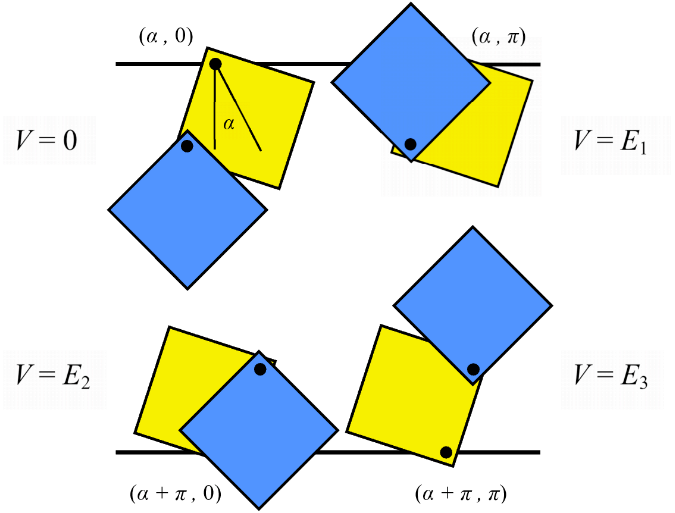
- Lagrangian $\mathcal{L} = T - V$:

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I_2\dot{\theta}_2^2$$

where $I_i = \frac{1}{6}m_i L^2$, and $V = V_0 + m_1 g y_1 + m_2 g y_2$

Equilibrium configurations

- $E_1 = \sqrt{2} m_2 g \ell$, $E_2 = E_1 \operatorname{cosec} \alpha$, $E_3 = E_1 + E_2$, with $\tan \alpha = m_2/(m_1 + m_2)$, $\ell = L - 2d$



Equations of motion

- Introduce

$$\varphi_1 = \theta_1 - \alpha, \quad \varphi_2 = \theta_2$$

- Euler-Lagrange leads to:

$$c_1\ddot{\varphi}_1 + c_2\left(\ddot{\varphi}_2 \sin\beta - \dot{\varphi}_2^2 \cos\beta\right) + c_3 \sin\varphi_1 = 0$$

$$c_4\ddot{\varphi}_2 + c_2\left(\ddot{\varphi}_1 \sin\beta + \dot{\varphi}_1^2 \cos\beta\right) + c_5 \sin\varphi_2 = 0,$$

with

$$c_1 = \frac{1}{12}\left[(m_1/m_2)L^2 + 3(m_1/m_2 + 2)\ell^2\right],$$

$$c_2 = \sqrt{2}\ell^2/4, \quad c_3 = E_2/4m_2,$$

$$c_4 = (L^2 + 3\ell^2)/12, \quad c_5 = E_1/4m_2,$$

$$\beta = \pi/4 + \alpha + \varphi_1 - \varphi_2$$

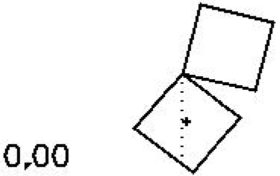
Behaviour with increasing energy

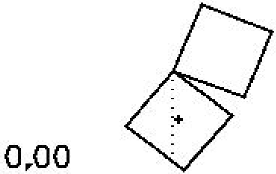
- Small amplitude motion: coupled linear oscillators

$$c_1\ddot{\varphi}_1 + \frac{1}{\sqrt{2}}c_2\left(\cos\alpha + \sin\alpha\right)\ddot{\varphi}_2 + c_3\varphi_1 = 0$$

$$c_4\ddot{\varphi}_2 + \frac{1}{\sqrt{2}}c_2\left(\cos\alpha + \sin\alpha\right)\ddot{\varphi}_1 + c_5\varphi_2 = 0$$

  - two strictly periodic normal modes
  - *slow mode*: co-oscillating (left)
  - *fast mode*: counter-oscillating (right)





Behaviour with increasing energy

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$$c_4\ddot{\varphi}_2 + \frac{1}{\sqrt{2}}c_2\left(\cos\alpha + \sin\alpha\right)\ddot{\varphi}_1 + c_5\varphi_2 = 0$$

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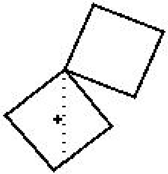
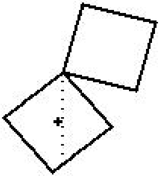
  - *fast mode*: counter-oscillating (right)

4,01



1,81



- Easy to demonstrate normal modes in real device
- Threshold energies for complete rotation
  - outer plate: $E > E_1$
  - inner plate: $E > E_2$
  - both plates: $E > E_3 = E_1 + E_2$
- High energy: device is a simple rotor
  - $T \gg V$
  - formally the same as setting g to zero
  - total angular momentum is conserved
  - motion is regular (two degrees of freedom and two constraints)
  - rotation in stretched configuration, with oscillation

Numerical solution: Methods

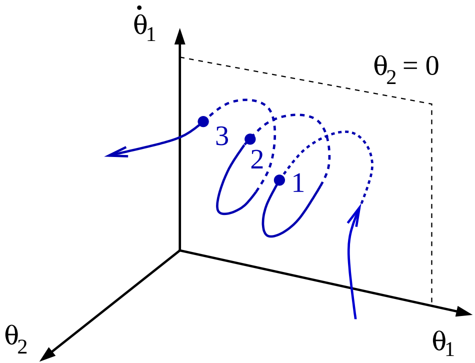
- Non-dimensionalization:

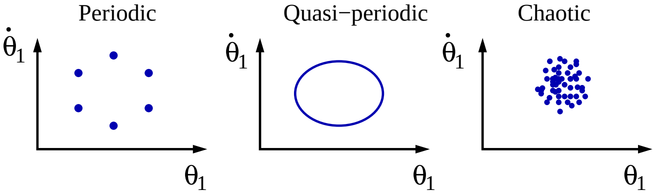
$$\overline{t} = \sqrt{g/L}t, \quad \overline{L} = \ell/L, \quad \overline{m} = m_1/m_2,$$

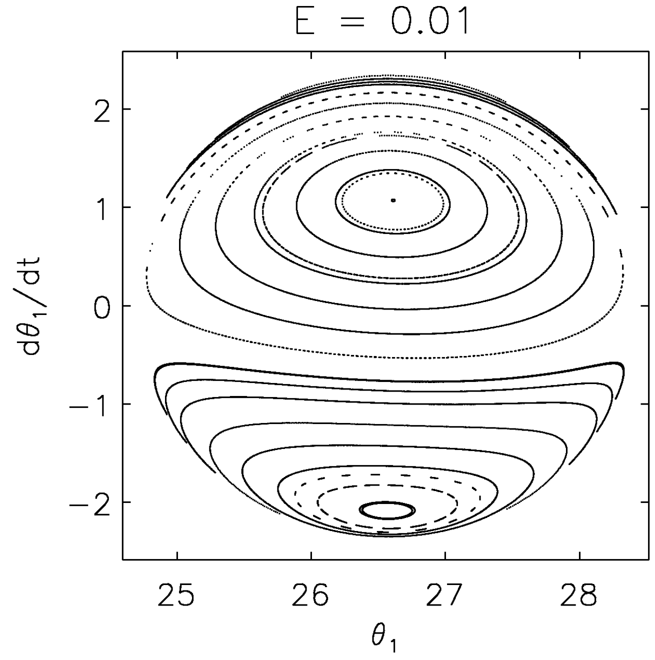
$$\overline{E} = \frac{E}{\frac{1}{12}m_2 g L}$$

  - thresholds: $\overline{E}_1 = 16.97$, $\overline{E}_2 = 37.95$, $\overline{E}_3 = 54.92$
- Equations solved with fourth order Runge-Kutta
  - conservation of energy used as a check on accuracy

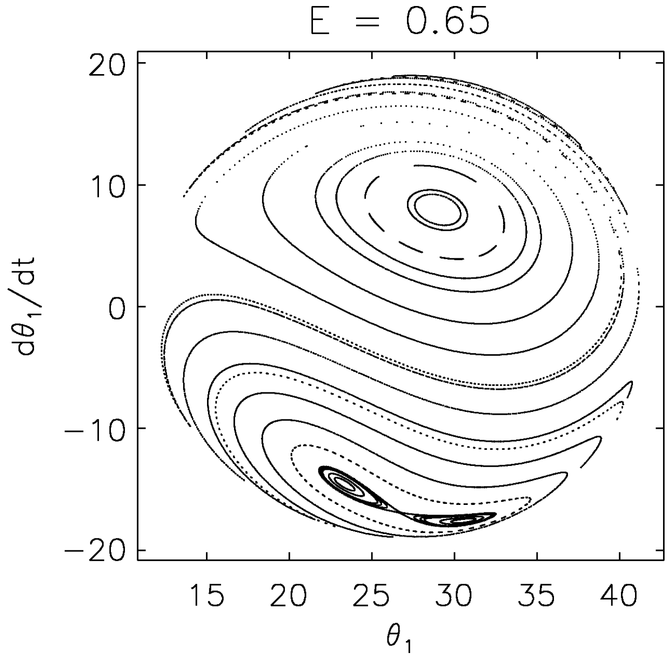
Poincare sections

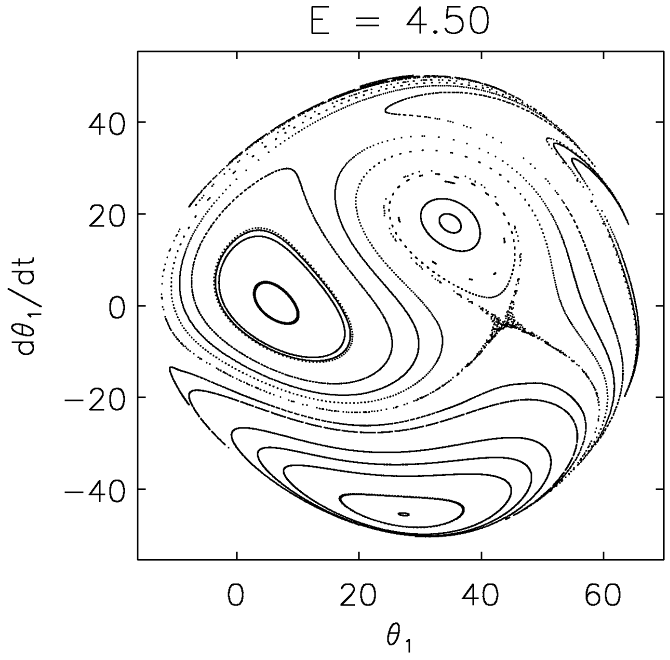


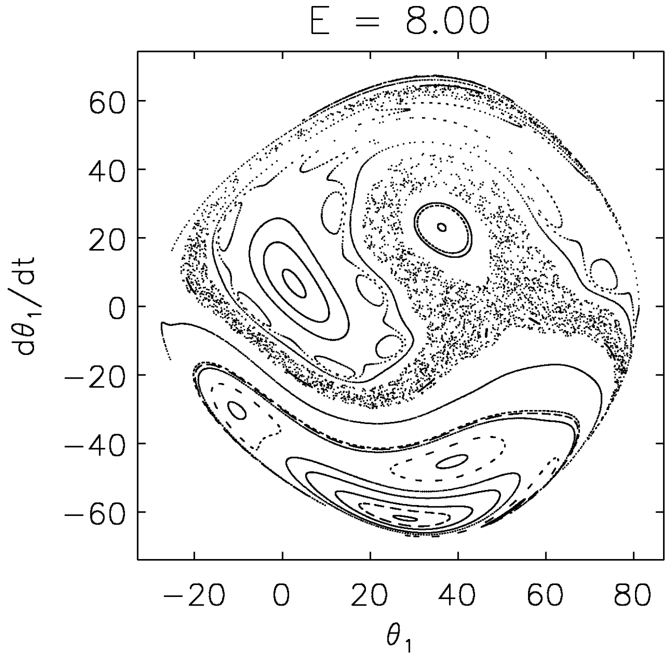


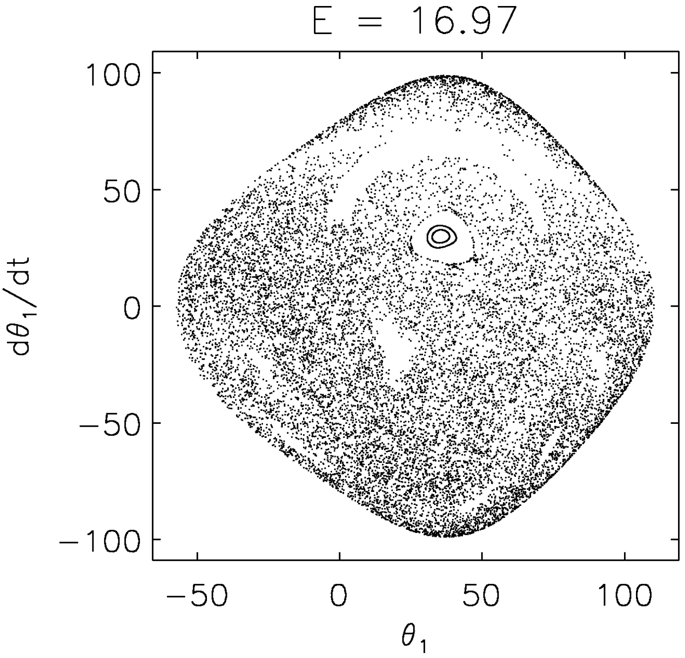


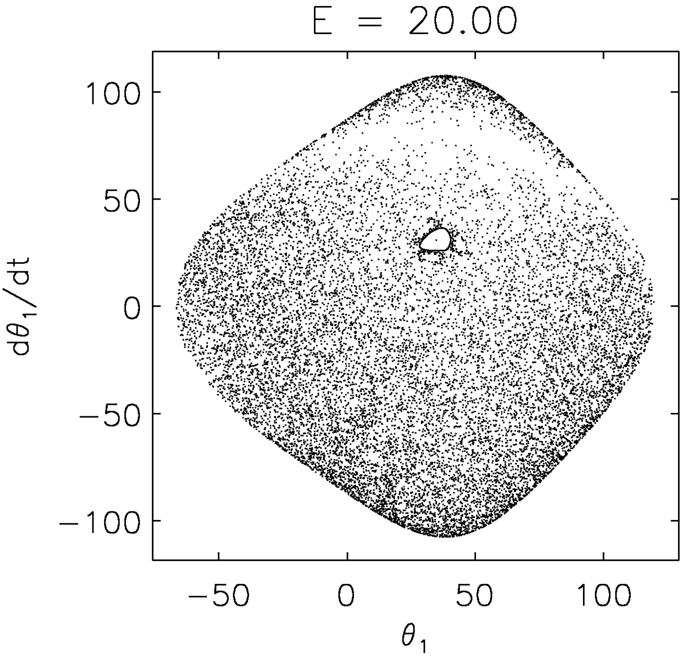
E = 0.01
2
1
0
−1
−2
dθ1/dt
25
26
27
28
θ1


E = 0.65
20
10
0
−10
−20
15
20
25
30
35
40
$d\theta_1/dt$
θ_1

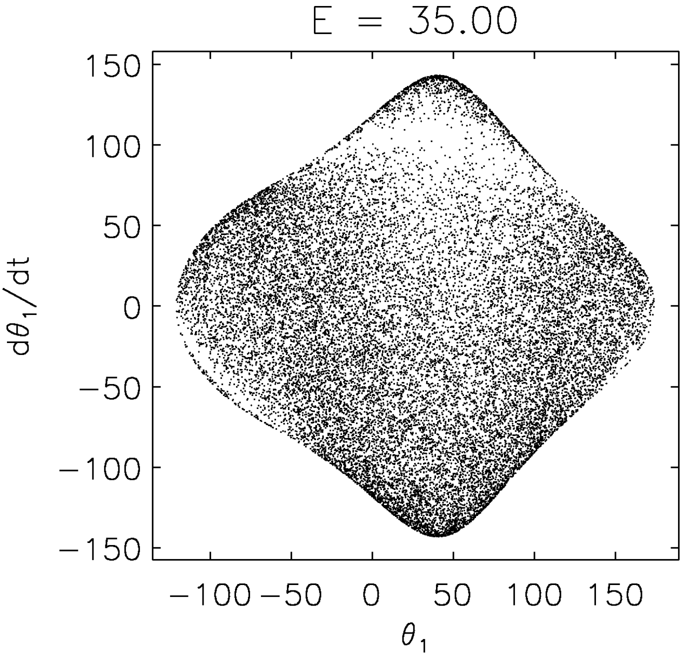

E = 4.50
40
20
0
−20
−40
0
20
40
60
dθ1/dt
θ1

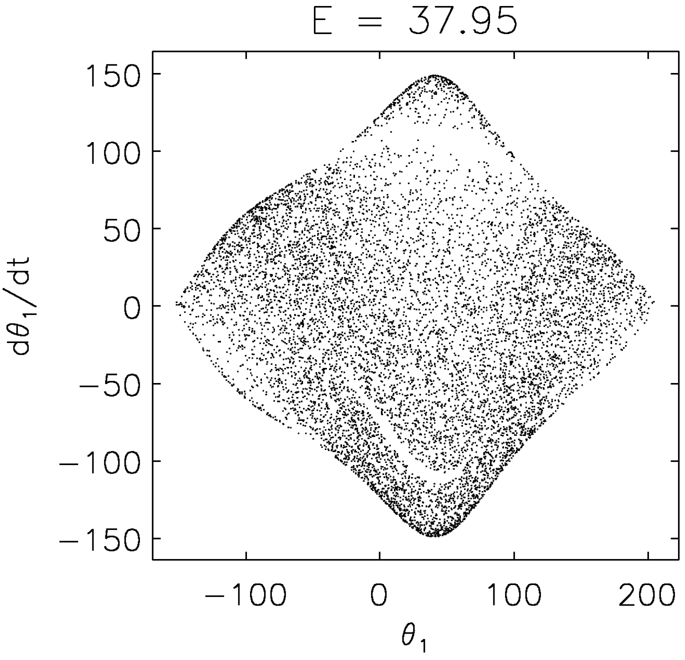

E = 8.00
60
40
20
0
−20
−40
−60
dθ1/dt
−20
0
20
40
60
80
θ1

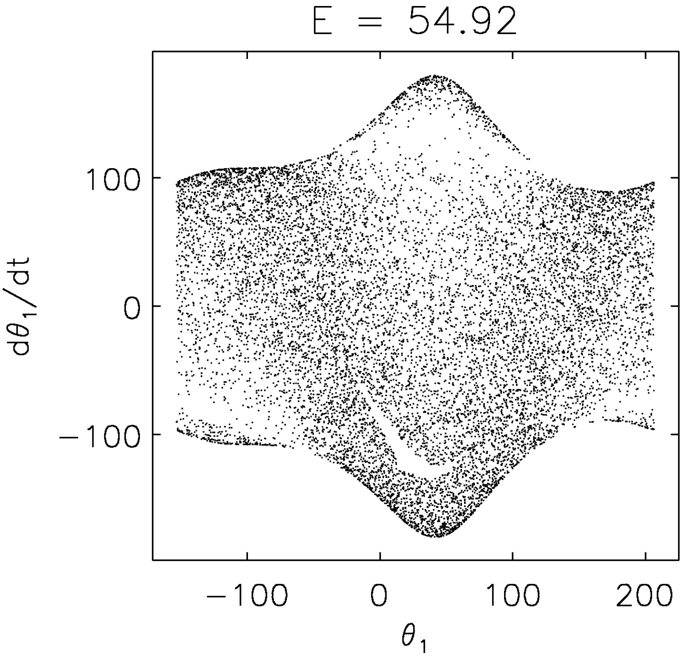

E = 16.97
100
50
0
−50
−100
dθ1/dt
−50
0
50
100
θ1

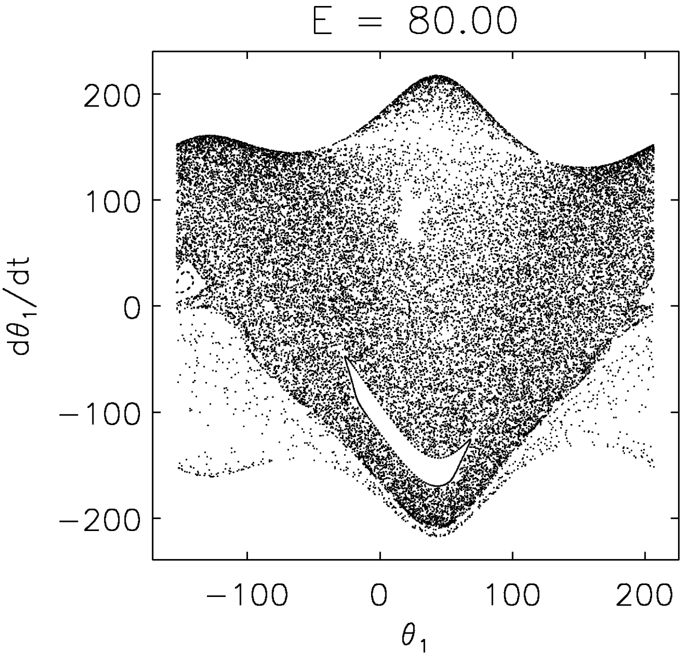


E = 20.00
100
50
0
−50
−100
dθ1/dt
−50
0
50
100
θ1

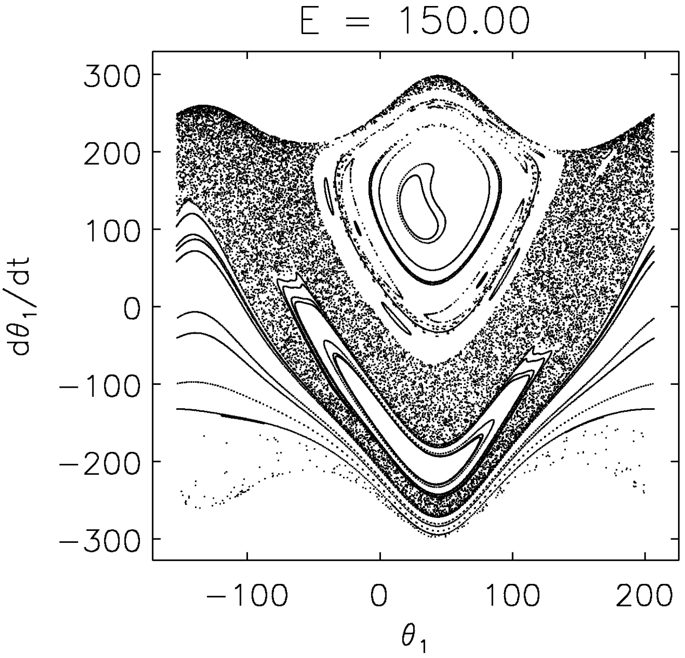

E = 35.00
150
100
50
0
−50
−100
−150
dθ1/dt
−100
−50
0
50
100
150
θ1


E = 37.95
150
100
50
0
−50
−100
−150
dθ1/dt
−100
0
100
200
θ1

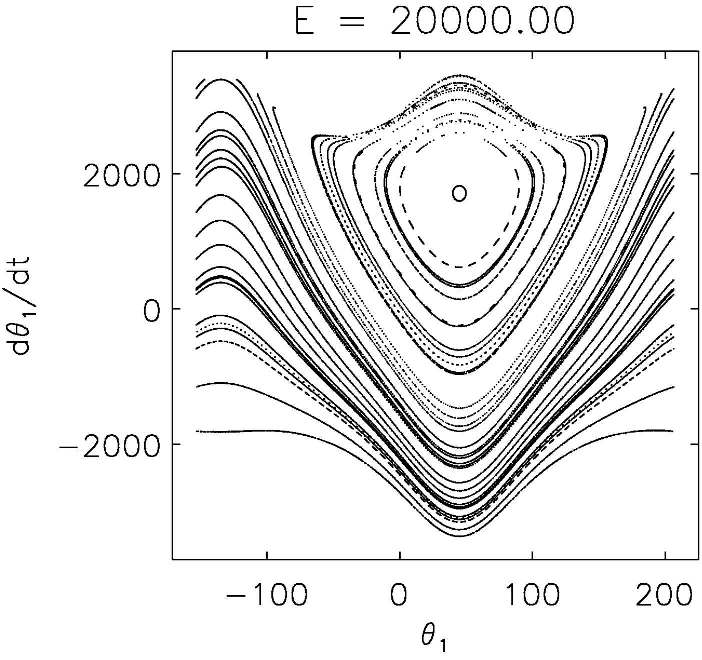

E = 54.92
100
0
−100
−100
0
100
200
$d\theta_1/dt$
θ_1



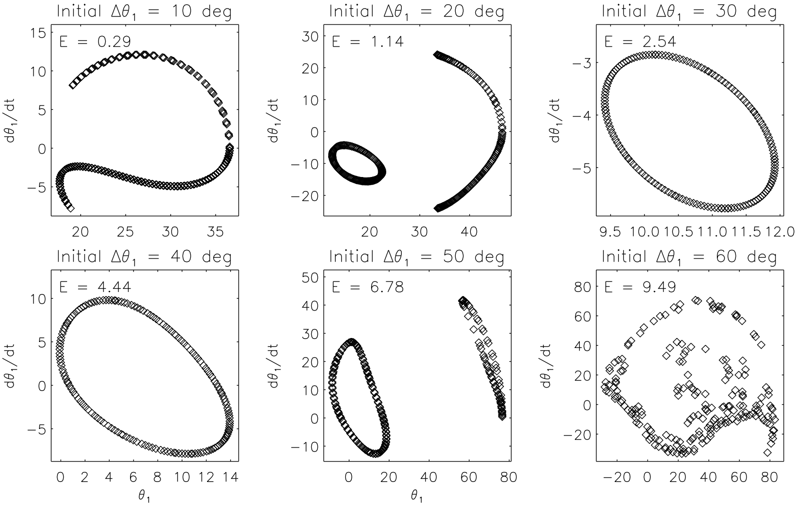
E = 80.00
200
100
0
−100
−200
dθ1/dt
−100
0
100
200
θ1



E = 150.00
300
200
100
0
−100
−200
−300
dθ1/dt
−100
0
100
200
θ1

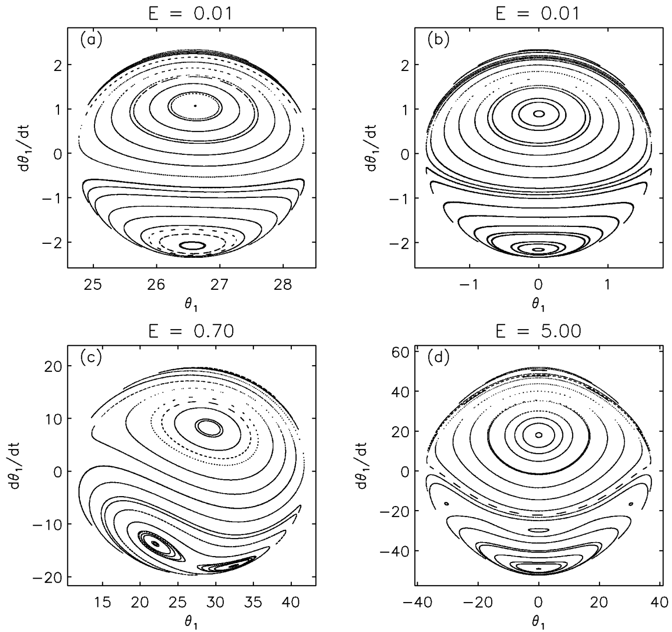


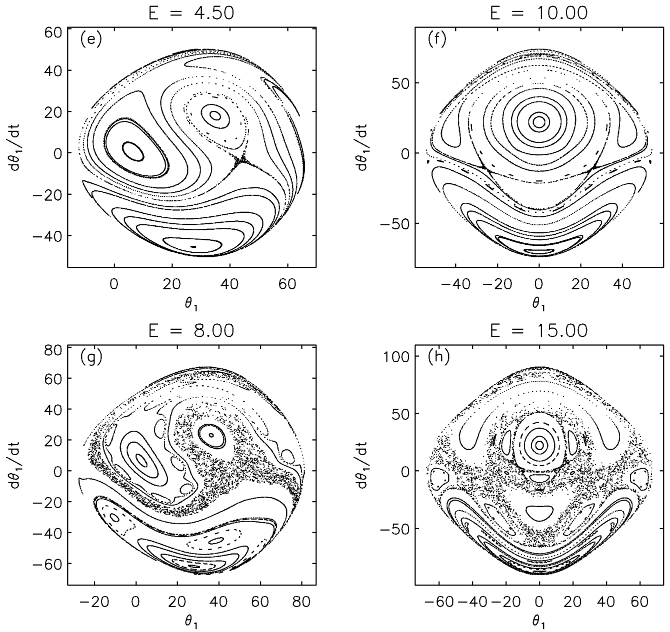
E = 20000.00
2000
0
−2000
−100
0
100
200
dθ1/dt
θ1



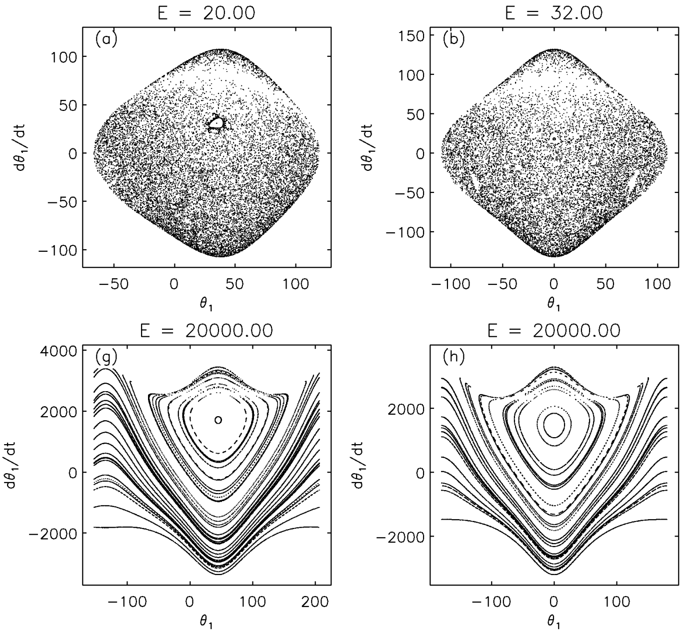
Initial Δθ1 = 10 deg
E = 0.29
Initial Δθ1 = 20 deg
E = 1.14
Initial Δθ1 = 30 deg
E = 2.54
Initial Δθ1 = 40 deg
E = 4.44
Initial Δθ1 = 50 deg
E = 6.78
Initial Δθ1 = 60 deg
E = 9.49
dθ1/dt
θ1

Comparison with simple double pendulum





E = 4.50
(e)
60
40
20
0
−20
−40
dθ1/dt
0
20
40
60
θ1
E = 10.00
(f)
50
0
−50
dθ1/dt
−40
−20
0
20
40
θ1
E = 8.00
(g)
80
60
40
20
0
−20
−40
−60
dθ1/dt
−20
0
20
40
60
80
θ1
E = 15.00
(h)
100
50
0
−50
dθ1/dt
−60
−40
−20
0
20
40
60
θ1


E = 20.00
(a)
dθ1/dt
θ1
E = 32.00
(b)
E = 20000.00
(g)
E = 20000.00
(h)

Experiments with the device

- Angle in repose: $\theta_1 = 15.5^\circ \pm 0.5^\circ$
  - simple model: $\alpha = \tan^{-1} \frac{1}{2} = 26.6^\circ$
  - the wheel is heavy!
- Timing normal modes:

$$T_\mathrm{f} = 0.70 \pm 0.01\,\mathrm{s} \quad T_\mathrm{s} = 1.23 \pm 0.01\,\mathrm{s}$$

  - simple model:

$$T_\mathrm{f} \approx 3.78\,(L/g)^{1/2} \quad T_\mathrm{s} \approx 8.03\,(L/g)^{1/2}$$

  evaluates to $T_\mathrm{f} \approx 0.64\,\mathrm{s}$, $T_\mathrm{s} \approx 1.36\,\mathrm{s}$ for $L = 0.28\,\mathrm{m}$

- Device is easy to set up in the $E = V_2$ equilibrium
  - may be released from this position multiple times
  - simple model implies chaos...

Summary

- School of Physics double square pendulum
  - a fun device!
  - useful for teaching dynamics
- Simple model investigated
  - equilibria, linear normal modes described
  - equations numerically solved
  - Poincare sections constructed
  - regular $\rightarrow$ chaotic $\rightarrow$ regular with increasing E
  - chaotic before outer plate rotates
  - slow mode persists
  - similar behaviour to simple double pendulum
  - illustrates scientific visualization
- Web pages:

http://www.physics.usyd.edu.au/∼wheat/sdpend/